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A Study on Topological Groups and Their Separation Axioms

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Abstract: **In this article we study topological groups and their separation axioms. The theory of topological groups is rich in terms of its own profound results and also in terms of its application. We study topological groups mainly to illustrate how the introduction of an algebraic structure on a space affects and enriches its topological properties. As the name implies, a topological group is a topological space whose underlying set is also endowed with a group structure. A topological group is a mathematical object with both an algebraic structure and topological structure. Topological group along with continuous group action are used to study continuous symmetries which have many applications.**

Keywords: **Filter base, interior of a set, Neighborhood system, semi topological group, separation axioms, and topological group.**

I. INTRODUCTION

This thesis is divided into five chapters. This chapter is a brief introduction and motivation to the discussion in the subsequent chapters. First we introduce semi topological group and some theorems related to semi topological group and neighborhood system of identity of semi topological group. In the second chapter we introduce topological groups and some conditions under which every semi topological group become topological group. In the third chapter we discuss about the special results involving separation axioms in topological groups. Here we are studying separation axioms because they acquire special character in case of topological groups.

II. TOPOLOGICAL GROUP

Motivation:

The theory of topological groups is rich both in terms of its own profound results and also in terms of its application. We study topological groups mainly to illustrate how the introduction of an algebraic structure on a space affects and enriches its topological properties. As the name implies, a topological group is a topological space whose underlying set is also endowed with a group structure. A topological group is a mathematical object with both an algebraic structure and topological structure. Topological groups along with continuous group action are used to study continuous symmetries which have many applications for example in physics.

Fundamentals of topology and group theory:

Topological spaces A set **X** with a family **u** of its subsets is called a topological space if the following conditions are satisfied-

X and ϕ are in **u**.

The intersection of any finite number of members of **u** is in **u**.

The arbitrary union of members of **u** is in **u**.

The members of **u** are called **u**-open sets of **X**. A topological space **X** with a topology **u** will be denoted by (X, u).

For any given set **X**, there always two topologies on **X** if (1) **u** consists of all subsets of **X**. this topology is known as discrete topology on **X**. (2) **u** consists of only **X** and ϕ. This topology is known as indiscrete topology on **X**. Let **u** and **v** be two topologies on a set **X**.

U is said to be **finer** than **v** or **u** $\supset$**v** if every **v**-open set is **u**-open if **u** $\supset$**v** then **v** is said to be coaser than **u**.

Interior of a set Let (X,u) be a topological space and **A** be any subset of **X**. The largest open set containing **A** is called interior of **A**. Clearly A of **X** is open iff $int(A) = A$.

Closed set The complement X-U of an open set U in a topological space **X** is called the **u** closed.

Neighborhood system:

Let (X,u) be a topological space. Let $x \in X$ a subset P of X is said to be a u-neighborhood of x if there exist a u-open set U such that $x \in U \subset P$. Observe that a neighborhood of a point $x \in X$ is not necessarily an open set. However, it is quite clear that an open set is a neighborhood of each point contained in it. The connection between the open set and neighborhood - A subset A of X is u-open iff for each $x \in A$ there exit a neighborhood Px of x such that $Px \subset A$.

For each $x \in Xu$, let Ux denote the totality of all u-neighborhood of x. then the following properties are immediately established by using the definition of neighborhood and open sets-

For each member U in Ux,$x \in U$.

If U is in Ux and W is any subset of X such that $U \subset W$, then W is in Ux.

Each finite intersection of sets in Ux is in Ux.

If U is in Ux, then there exist a V in Ux such that $V \subset U$ and $Ux \in Uy$ for each $y \in V$ where Uy is the totality of all u-neighborhood of y.

Definition:- Bases:- Let $(X, \mathfrak{I})$ be a topological space. A subfamily B of $\mathfrak{I}$ is said to be a base for $\mathfrak{I}$ if every member of $\mathfrak{I}$ can be expressed as the union of some members of B.

For example, in a metric space every open set can be expressed as a union of open balls.

Let $(X, \mathfrak{I})$ be a topological space and $B \subset \mathfrak{I}$. Then B is a base for $\mathfrak{I}$ iff for any open set G containing x, there exists $B \in B$ such that $x \in B$ and $B \subset G$.

Subbase:

A family S of subsets of X is said to be a sub-base for a topology $\mathfrak{I}$ on X if the family of all finite intersection of members of S is a base for $\mathfrak{I}$. Any base for a topology is also a sub-base for the same.

Filter:

A filter on a set X is a non-empty family F of subsets of X such that (i)$\Phi \notin F$, (ii) F is closed under finite intersection, and (iii) if $B \in F$ and $B \subset A$ then $A \in F$ for all $A, B \subset X$.

Filter base:

Let F be a filter on a set X. Then a subfamily B *of* F is said to be a filter base if for any $A \in F$ there exists $B \in B$ such that $B \subset A$.

Count ability axioms:

Local base- Let x be a point in a topological space X and N_X the neighborhood system of x. A subfamily B of N_X is said to be a local base at x if for each $U \in N_X$ there is a member B of B such that $B \subset U$.

First axiom of countability-A topological space X is said to satisfy the first axiom of countability if each point x in X has a countable local base.

Second axiom of Countability- A topological space X is said to satisfy the second axiom of countability if the topology $\mathfrak{I}$ has a countable base.

Compact spaces-A topological space $(X, \mathfrak{I})$ is called a compact space if every open cover for X has a finite subcover.

Locally compact-A topological space $(X, \mathfrak{I})$ is said to be locally compact if each point x in X has at least one compact neighborhood.

Semi topological group and Topological group:

Definition

A topological space G that is also a group is called a **semi topological group** if the mapping

$g_1 : (x,y) \rightarrow xy$

Of $G \times G$ onto G is continuous in each variable separately.

A topological space G that is also a group is called a **topological group** if the mapping g_1 is continuous in both the variable and if the inverse mapping

$g_2 : x \rightarrow x^{-1}$

of G onto G is continuous in each variable.

If the group operation is addition instead of multiplication xy and x^{-1} should be regarded as $x + y$ and $-x$ respectively.The identity of a multiplicative group will be denoted by e and that of additive group by 0.

Examples of topological group:

1. The real number with respect to addition with its usual topology form a topological group.

2. The non-zero real number or the non-zero complex number form a topological group with respect to multiplication.

3. All euclidean spaces under the usual addition are topological group.

4. Any group given the discrete or indiscrete topology form a topological group.

5. If $\{Gi : i \in I\}$ is a collection of topological group, the product space $\prod Gi$ can be made into a topological group under co-ordinate wise multiplication.

6. In every banach algebra with multiplicative identity the set of invert- ible elements form topological group under multiplication Euclidean n-space R^n with addition and standard topology form a topological group.

If we put $UV = \{xy : x \in U, y \in V\}$ and $U^{-1} = \{x^{-1} : x \in U\}$, where U and V are subsets of a group G, and in additive case, $U + V = \{x + y : x \in U, y \in V\}$, $-U = \{-x : x \in U\}$, then the mappings g_1 and g_2 can be expressed as follows-

g1 is continuous in x(or y) iff for each neighborhood W of xy there exits a neighborhood U (or V) of x(or y) such that Uy subset W(or xV subset W). Moreover g_1 is continuous in both x and y iff for each neighborhood W of xy there exists a neighborhood U of x and V of y such that UV subset W. Similarly g_2 is continuous if and only of for each neighborhood W of x^{-1}, there exists a neighborhood U of x such that $U^{-1} \subset W$.

Theorem 1

Every topological group is a semi topological group. But the converse is not true.

Solution:

First part is almost clear. Since it is a topological group therefore it satisfy both the conditions 1.1 and 1.2. Again since condition 1.1 is satisfied therefore it is a semi topological group.

To prove the converse part let us take G=R, the real line as an additive abelian group. Let G be endowed with a topology with $\{[a, b) : -\propto < a < x < b < \propto\}$, the system of left closed and right open interval as its base. Since for each neighborhood $[a, b)$ of the identity 0, $[a, b/2)$ is also a neighborhood of 0, it follows that the mapping $g1$ is continuous in both variables together at 0. Therefore G is a semi topological group.

But the inverse mapping $g_2 : x \rightarrow -x$ is not continuous at 0.because if $[0, b)$ is a neighborhood of 0 then there exist no neighborhood V of 0 such that $-V \subset [0, b)$.therefore g_2 is not continuous. Hence G is not a topological group.

It shows that every topological group need not be a topological group.

Therefore the real line $\Re$, with the topology $\{[a, b) : -\infty < a < x < b < \infty\}$,the system of left closed and right open interval as its base, is an example of a semi topological group which is not a topological group.

Theorem 2

Let a be a fixed element of a semi topological group G Then the mapping

$r_a : x \rightarrow xa \quad l_a : x \rightarrow ax$

of G onto G are homeomorphism of G.

proof:-

It is clear that r_a is one-one and onto mapping. Let W be a neighborhood of xa. Since G is a semi topological group, there exists a neighborhood U of x such that $Ua \subset W$. Therefore any neighborhood W of $r_a(x)$, there exists a neighborhood U of x such that $r_a(U) \subset W$. This shows that r_a is continuous. Moreover, it is easy to see that the inverse $r^{-1} : x \rightarrow xa^{-1}$, is continuous by the same argument as above. Hence r_a is a homeomorphism. Similarly the mapping l_a is also homeomorphism.

r_a *and* l_a are respectively called the right and left translation of G.

Corollary 1

Let F be a closed, P an open, and A any subset of a semi topological group G and let $a \in G$.

Then:

Fa,aF are closed.

Pa,aP,AP and PA are open.

proof:-

Since the mapping $r_a : x \rightarrow xa$ and $l_a : x \rightarrow ax$ are homeomorphisms.

Therefore since F is closed, Fa and aF are also closed.

By the same argument, since P is open, Pa, aP are also open. Now since $AP = \cup aP$, $PA = \cup Pa$, and we know that union of open set is open. Therefore AP and PA is also open.

Corollary 2

Let G be a semi topological group. For any $x_1, x_2 \in G$, there exists a homeomorphism f of G such that $f(x_1) = x_2$.

proof:-

Let $x_1^{-1}x_2 = a \in G$, and consider the mapping $f : x \rightarrow xa$. Then by theorem 2 we know that f is a homeomorphism and

$f(x_1) = x_1 a$

$= x_1(x_1^{-1} * x_2)$ 1

$= (x_1 * x_1^{-1})x_2$

$= x_2$

Thus there exists a homeomorphism f of G such that $f(x_1) = x_2$.

Every group can be converted into a topological group by considering it with the discrete or indiscrete topology. Thus every group can be made into a topological group.

But every topological space cannot be converted into topological group. Let $(G, ., \Im)$ be a topological group. For $a \in G$, define $La: G \rightarrow G$ and $Ra: G \rightarrow G$ by $La(x) = ax$, $Ra(x) = xa$ for $x \in G$. These functions are called respectively the right and left translation by a and by theorem 2 these functions are homeomorphism. Unless a is the identity element of G ,

L_a *and* R_a have no fixed points. Thus we see that the space $(G, \mathfrak{I})$ cannot have the fixed point property, unless G consists of only one point. For example, that the unit interval cannot be made into a topological group.

Neighborhood system of identity of a semi topological group:

From theorem 2, it follows that if one knows a fundamental system of neighborhood of the identity of a semi topological group, then one can find a fundamental system of neighborhoods of any other point by translation.

Theorem 3

If $\{U\}$ is a fundamental system of open neighborhood of e in a semi topological group G, then $\{xU\}$ and $\{Ux\}$, where x runs over G and U over $\{U\}$, form bases of the topology of G.

Conversely, let a filter base $\{U\}$ be given so that each U contains e and for each U and each $x \in U$ there exists V and W in $\{U\}$ such that $xV \subset U$ and $Wx \subset U$. Then there exixt a topology u on G ,endowed with u, is a semi topological group.

proof:-

Let $a \in G$ and let W be an open neighborhood of a.

Since $l_a : x \to a^{-1}x$ is a homeomorphism (by theorem2). Therefore $l_a(W) = a^{-1}W$ is an openset containing e, W is an open set containing a. Again since $\{U\}$ is a fundamental system of open neighborhood of e in a semi topological group G therefore there exists a U in $\{U\}$ such that $U \subset a^{-1}W$. This implies $aU \subset W$, which proves that $\{xU\}$ is a base of the topology on G. Similarly we can show that $\{Ux\}$ is also a base.

Conversely, let U denote the family of all finite intersection of members in $\{U\}$. Then U is a non-empty family of U, each of which contains e. Furthermore, for any $U_i = \cap U_i, 1 \le i \le n$, $xU_i = x(\cap U_i) = \cap xU_i$, for any $x \in G$. And if $x \in \cap U_i$, then there exists a V_i for each $i, 1 \le i \le n$, such that $xV_i \subset U_i$ and, this shows that the family U also satisfies the conditions assumed for the filter base $\{U\}$. By the definition of a sub base, the family of finite intersection of the family $\{xU\}$, where x runs over G and U over $\{U\}$ forms a base of the topology u on G. Now if $y \in \cap x_i U_i$, then $y \in U_i$ for each i, hence by assumption, there exists a $V_i \in \{U\}$ such that $x^{-1}yV_i \subset U_i$, *or* $yV_i \subset x_i U_i$, this shows that $y(\cap V_i) = \cap(yV_i) \subset \cap x_i V_i$. Therefore $\{yU\}$ forms a fundamental system of open neighborhood of y. Similarly we can show that $\{Uy\}$ also forms a fundamental system of neighborhood of y.

Now to complete the proof, G endowed with u is a semi topological group.

Consider the mapping $g_1 : (x,y) \longrightarrow$ xy.

Assume that x is fixed, and let U_i be any member of U. Then xyU_i is a member of fundamental system of neighborhood of xy. Since $y \in yU_i$ and yU_i is a u-neighborhood of y, $x(yU_i) \subset xyU_i$, this proves the continuity of g_1 in y while x is kept fixed.

Embedding of any group in a product group:

If for each $\alpha \in A$, any index set, the product $\prod G_\alpha$ is denoted simply by G^A.Clearly, G^A is the set of all mappings of A into G. If G is a group then G^A is also a group. Now if $A = G$, which is a group, then G^G is also a group.

Theorem 4

Any group G can be embedded into G^G that is there exists a one-to- one mapping of G onto a subset of G^G.

proof:-

For each $a \in G$, *let* r_a *denote the right translation of G onto G that is* $r_a : x \to xa$, clearly $r_a \in G^G$. Define $\eta_r : a \to r_a$, then η_r is one-one mapping. since

$\eta a = \eta b$

$ra(x) = rb(x)$

$xa = xb, \forall x$

$a = b$

Thus G is mapped onto $\eta r\ (G) \subset G^G$.In other words, G can be identified with the set of all its right translation. Similarly, one can identify G with the set of all its left translation as well.

Notations:-

The mapping $\eta r\ : a \to ra$ and $\eta r\ : a \to la$ of G onto G^G will be called the right and left canonical embedding of G into G^G respectively. In case G is the abelian group $ra = la$ and thus $\eta r = \eta l$.

σ-topologies and semi topological group:

Let u and v be two topologies on a set E. Let V_X denote a fundamental system of v-neighborhood of an arbitrary point $x \in E$. Let $cl_u V_X$ denote the u-closure of V_X. Let $u(v)$ denote the topology on E which has $cl_u V_X$ as a fundamental system of neighborhood of x, where V_X runs over V_X and x over E. Since for each $cl_u V_X$ in $cl_u V_X$, $cl_u V_X \supset V_X$, the topology u(v) is coarser than v. In symbols $u(v) \subset v$.

Theorem 5

If E_v is a regular topological space and u any other topology on E such that $u \supset v$ then $v = u(v)$

proof:- Let V_X be a fundamental system of v-neighborhood of an arbitrary point $x \in E$. Since E_v is regular, it can be assumed that each V_X is v-closed. But then $u \supset v$ implies that each V_X is u-closed as well and therefore $cl_u V_X = V_X$ for each V_X in V_X.this proves that $u(v) = v$

B-type and C-type semi topological group:

Definition:

(a)B-type semi topological group-A topological group G_u is said to be of B-type if for the topology c on G induced from $C(G, G) \subset G^G$, $c \supset u$ and $u(c) = c$ imply $c = u$.

Some conditions under which every semi topological group is a topological group

Every normal semi topological group G_u of B-type is a topological group.

Every regular baire semi topological group G_u satisfying the second axiom of countability is of B-type and hence a topological group.

A complete metric and separable semi toplogical group is a topological group.

A regular locally compact semi topological group satisfying the second axiom of countability is a topological group.

A metric compact semitopological group is a topological group.

A Hausdorff compact semi topological group satisfying the second axiom of countability is a topological group.

Separation axiom on topological group:

Let X_u be a topological space. the following separation axioms in X_u are known as

T_0 space- A topological space X is called a $T0$ space if for each pair of distinct points x and y in X there exists an open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

T_1 space- A topological space X is called a $T1$ space if for each pair of distinct points x and y in X there exists open sets U and V such that $x \in U$ and $y \notin U$ and $y \in V$ and $x \notin V$.

T_2 space- A topological space X is called a $T2$ space (or Hausdorff space) if for each pair of distinct points x and y in X there exists an open neighborhood U of x and an open neighborhood V of y such that $U \cap V = \phi$

Every $T2$ space is a $T1$ space and every $T1$ space is $T0$ space.

Regular spces- A topological space $(X, \mathfrak{J})$ is said to be regular if for each point $x \in X$ and for each closed set F in X not containing x, there exists an open neighborhood U of x and an open neighborhood V of F such that $U \cap V = \phi$

$T3$ space- A regular $T1$-space is called $T3$ space. Every $T3$ space is a T_2 space.

Normal spaces- A topological space $(X, \mathfrak{J})$ is called a normal space if for each pair of disjoint closed subset $F1$ and $F2$ in X there exists an open neighborhood $U1$ of $F1$ and open neighborhood $U2$ of $F2$ such that $U1 \cap U2 = \phi$

$T4$ space- A normal $T1$ space is called $T4$ space. Every discrete space is a $T4$ space.

Every normal space need not be regular.

Completely regular- let C be any closed subset of X_u and x be any pt of X such that $x \notin C$ then there exists disjoint open sets U and V such that $x \in U, C \subset V$. Then there exists real valued continuous function on X on [0, 1] such that f(x)=0, f(1)=1.

Separation axioms in topological group:

Theorem 6

For a topological group G, the following statements are equivalent:

G is a $T0$ space.

G is a $T1$ space.

G is a $T2$ or Hausdorff space.

$\cap U = e$, where U is a fundamental system of neighborhood of e.

proof-

We shall show that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$

For $(a) \Rightarrow (b)$, that is let G be a $T0$ space, let $x \neq y$, x, $y \in G$ .By (a) for at least one (say,x) of x and y, there exists an open neighborhood P of x such that $y \notin P$. since $x^{-1}P =$ is an open neighborhood of e, $V \cap V^{-1} = Q$ is

an open symmetric neighborhood of e and therefore yQ is a neighborhood of y. Now x ∈ yQ

because if $x \in yQ$, then we get $x^{-1} \in Qy^{-1}$

and hence $x^{-1} \in Qy^{-1} \subset Vy^{-1} \subset x^{-1}Py^{-1}$. But from this we get that $e = xx^{-1} \in xx^{-1}Py^{-1} = Py^{-1}$, or $y \in P$,which is a contradiction. therefore we get a open neighborhood P of x such that $y \notin P$ and a neighborhood yQ of y such that x∉ yQ. Therefore G is a T1 space.

For $(b) \Rightarrow (c)$, that is given that G is a $T1$ space. Now let $x \neq y$, x, $y \in G$,

x is a closed set and therefore $P = G - x$ is an open neighborhood of ~~y and~~ hence $y^{-1}P$ is an open neighborhood of e. Let V be an open neighborhood of e such that $VV^{-1} \subset y^{-1}P$. Then yV is an open neighborhood of y. Let $Q = G - yV$ which is an open set, and $x \in Q$. For otherwise $x \in yV$ and hence $xV \cap yV \neq \phi$.

But this shows that $x \in yVV^{-1} \subset y(y^{-1}P) = P$, which is a contradiction.

because $x \notin P$. Clearly $Q \cap yV = \phi$, $y \in yV$,and x,yV and Q are open sets. This proves that G is a T_2 space.

For $(c) \Rightarrow (d)$ let $x \in U$ for each U in {U} and assumes $x \neq e$. Then (c) implies that there exists a neighborhood P of e such that $x \notin P$. But then there exists a U in {U} such that $U \subset P$. We have the contradiction $x \in U \subset P$ and $x \notin P$. Hence $x = e$ and (d) is established.

For $(d) \Rightarrow (a)$, let $x \neq y$ then $xy^{-1} \neq e$ and hence, by (d) there exists a U in U such that $xy^{-1} \notin U$. Thus Uy being the neighborhood of y an $x \notin Uy$, (a) is proved.

III. CONCLUSION

In this thesis we make a study of semi topological group and topological group. Here we give some basic theorems concerning semi topological group and topological group. Here we seen that a topological group is always a semi topological group but the converse is not always true as shown by examples. Here we also give some conditions under which every semi topological group is also a topological group. We discuss the separation axiom in topological group because they acquire special character in case of topological group.

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